Exercise 10

Compute AB, $\det A$, $\det B$, $\det(AB)$, and $\det(A+B)$ for

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Solution

Compute the product of A and B and their respective determinants.

$$\begin{aligned} AB &= \begin{bmatrix} 3 & 0 & 1 \\ 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} (3)(1)+(0)(2)+(1)(0) & (3)(0)+(0)(0)+(1)(1) & (3)(-1)+(0)(1)+(1)(0) \\ (1)(1)+(2)(2)+(-1)(0) & (1)(0)+(2)(0)+(-1)(1) & (1)(-1)+(2)(1)+(-1)(0) \\ (1)(1)+(0)(2)+(1)(0) & (1)(0)+(0)(0)+(1)(1) & (1)(-1)+(0)(1)+(1)(0) \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1 & -3 \\ 5 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \det A &= \begin{vmatrix} 3 & 0 & 1 \\ 1 & 2 & -1 \\ 1 & 0 & 1 \end{vmatrix} \\ &= 3\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} \\ &= 3[(2)(1)-(-1)(0)] - 0[(1)(1)-(-1)(1)] + 1[(1)(0)-(2)(1)] \\ &= 3(2) - 0(2) + 1(-2) \\ &= 4 \end{aligned}$$

$$\begin{aligned} \det B &= \begin{vmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \\ &= 0\begin{vmatrix} 0 & -1 \\ 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} \\ &= 0[(0)(1)-(-1)(0)] - 1[(1)(1)-(-1)(2)] + 0[(1)(0)-(0)(2)] \\ &= 0(0) - 1(3) + 0(0) \\ &= -3 \end{aligned}$$

Compute the determinant of AB and the determinant of $A+B$.

$$
\begin{aligned}
\det(AB) &= \begin{vmatrix} 3 & 1 & -3 \\ 5 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \\
&= 3\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 5 & 1 \\ 1 & -1 \end{vmatrix} - 3\begin{vmatrix} 5 & -1 \\ 1 & 1 \end{vmatrix} \\
&= 3[(-1)(-1)-(1)(1)] - 1[(5)(-1)-(1)(1)] - 3[(5)(1)-(-1)(1)] \\
&= 3(0) - 1(-6) - 3(6) \\
&= -12
\end{aligned}
$$

$$
\begin{aligned}
A + B &= \begin{bmatrix} 3 & 0 & 1 \\ 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 4 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\det(A+B) &= \begin{vmatrix} 4 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{vmatrix} \\
&= 4\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} - 0\begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} \\
&= 4[(2)(1)-(0)(1)] - 0[(3)(1)-(0)(1)] + 0[(3)(1)-(2)(1)] \\
&= 4(2) - 0(3) + 0(1) \\
&= 8
\end{aligned}
$$